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# On the Schur Test for $L_2$-Boundedness of Positive Integral Operators with a Wiener–Hopf Example

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tion 1.2, about the central role played by $L_2$-spaces in the general theory of these operators.

Suppose $(\Omega, \mathfrak{M}, \mu)$ is a measure space and that $K: \Omega \times \Omega \to [0, \infty)$ is an $\mathfrak{M} \times \mathfrak{M}$-measurable kernel. The special case of Proposition 1.2 for symmetrical kernels says that such a linear integral operator is bounded on *any* reasonable normed linear space $X$ of $\mathfrak{M}$-measurable functions only if it is bounded on $L_2(\Omega, \mathfrak{M}, \mu)$ where its norm is no larger. The general form of Schur's condition (Halmos and Sunder "Bounded Integral Operators on $L_2$-Spaces," Springer-Verlag, Berlin/New York, 1978) is a simple corollary which, in the symmetrical case, says that the existence of an $\mathfrak{M}$-measurable (not necessarily square-integrable) function $h > 0$ $\mu$-almost-everywhere on $\Omega$ with

$$\mathscr{K} h(x) = \int_\Omega K(x, y)\, h(y)\, \mu(dy) \leqslant \Lambda h(x) \qquad (x \in \Omega) \tag{$*$}$$

implies that $\mathscr{K}$ is a bounded (self-adjoint) operator on $L_2(\Omega, \mathfrak{M}, \mu)$ of norm at most $\Lambda$. When $(\Omega, \mathfrak{M}, \mu)$ is $\sigma$-finite, we show that Schur's condition is sharp: in the symmetrical case the boundedness of $\mathscr{K}$ on $L_2(\Omega, \mathfrak{M}, \mu)$ implies, for any $\Lambda > \|\mathscr{K}\|_2$, the existence of a function $h \in L_2(\Omega, \mathfrak{M}, \mu)$ which is positive $\mu$-almost-everywhere and satisfies $(*)$.

Such functions $h$ satisfying $(*)$, whether in $L_2(\Omega, \mathfrak{M}, \mu)$ or not, will be called *Schur test functions*. They can be found explicitly in significant examples to yield best-possible estimates of the norms for classes of integral operators with non-negative kernels. In the general theory the operators are not required to be symmetrical (a theorem of Chisholm and Everitt (*Proc. Roy. Soc. Edinburgh Sect. A* **69** (14) (1970/1971), 199–204) on non-self-adjoint operators is derived in this way).

They may even act between different $L_2$-spaces. Section 2 is a rather substantial study of how this method yields the exact value of the norm of a particular operator between different $L_2$-spaces which arises naturally in Wiener–Hopf theory and which has several puzzling features. 



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## 1. $L_2$-SPACES IN THE THEORY OF POSITIVE INTEGRAL OPERATORS AND THE SCHUR TEST

The main results of this section are Propositions 1.4 and 1.5. We begin with some general theory to set the scene. (See the monograph by Halmos and Sunder [2] for an extensive treatment of integral operators on $L_2$-spaces).

Suppose that $(\Omega, \mathfrak{M}, \mu)$ and $(\Gamma, \mathfrak{N}, \nu)$ are two measure spaces, that $K: \Gamma \times \Omega \to [0, \infty)$ is measurable on the product space. Let $\mathscr{D}(\mathscr{K})$ be the linear space of $\mathfrak{M}$-measurable functions $u$ on $\Omega$ for which

$$\mathscr{K} u(x) = \int_\Omega K(x, y)\, u(y)\, d\mu_y$$

is finite $\nu$-almost-everywhere. Similarly, let $\mathscr{D}(\mathscr{K}^\dagger)$ be the set of $\mathfrak{N}$-measurable functions $v$ on $\Gamma$ such that

$$\mathscr{K}^\dagger v(y) = \int_\Gamma v(x)\, K(x, y)\, d\nu_x$$

is finite $\mu$-almost-everywhere, and let $\mathscr{D}_2(\mathscr{K}) = \{u \in L_2(\Omega, \mathfrak{M}, \mu) \cap \mathscr{D}(\mathscr{K}):$ $\mathscr{K} u \in L_2(\Gamma, \mathfrak{N}, \nu)\}$. We begin with the observation that $\mathscr{K}$ (similarly $\mathscr{K}^\dagger$) is often closed.

PROPOSITION 1.1. *Suppose that $L_2(\Omega, \mathfrak{M}, \mu) \subset \mathscr{D}(\mathscr{K})$. Then the linear operator $\mathscr{K}$ with domain $\mathscr{D}_2(\mathscr{K})$ is a closed linear operator on $L_2(\Omega, \mathfrak{M}, \mu)$. Consequently, if $L_2(\Omega, \mathfrak{M}, \mu) = \mathscr{D}_2(\mathscr{K})$, then $\mathscr{K}: L_2(\Omega, \mathfrak{M}, \mu) \to L_2(\Gamma, \mathfrak{N}, \nu)$ is bounded.*

*Proof.* Let $\{u_n\} \subset \mathscr{D}_2(\mathscr{K})$ be such that

$$u_n \to u \text{ in } L_2(\Omega, \mathfrak{M}, \mu) \qquad \text{and} \qquad \mathscr{K} u_n \to v \qquad \text{in } L_2(\Gamma, \mathfrak{N}, \nu).$$

We must show that $u \in \mathscr{D}_2(\mathscr{K})$ and that $\mathscr{K} u = v$. Without loss of generality, we may suppose that $\|u_n - u\|_{L_2(\Omega, \mathfrak{M}, \mu)} \leqslant 1/2^n$ and let

$$0 \leqslant f = \sum_{n=1}^{\infty} n\, |u_n - u| \qquad \text{in } L_2(\Omega, \mathfrak{M}, \mu).$$

Then for each $n$,

$$-f(y) \leqslant n(u_n(y) - u(y)) \leqslant f(y) \qquad \mu\text{-almost-everywhere.}$$

Since $f \in L_2(\Omega, \mathfrak{M}, \mu) \subset \mathscr{D}(\mathscr{K})$ and $K \geqslant 0$,

$$-\infty < -\mathscr{K}f(x) \leqslant n(\mathscr{K}u_n(x) - \mathscr{K}u(x)) \leqslant \mathscr{K}f(x) < \infty$$
$$\nu\text{-almost-everywhere.}$$

Therefore

$$\mathscr{K}u_n(x) - \mathscr{K}u(x) \to 0 \qquad \nu\text{-almost-everywhere.}$$

However, since $\mathscr{K}u_n \to v$ in $L_2(\Gamma, \mathfrak{N}, \nu)$, a subsequence converges pointwise $\nu$-almost-everywhere to $v$. Therefore $\mathscr{K}u = v$ and $u \in D_2(\mathscr{K})$. Thus $\mathscr{K}$, with domain $\mathscr{D}_2(\mathscr{K})$, is a closed linear operator on $L_2(\Omega, \mathfrak{M}, \mu)$ into $L_2(\Gamma, \mathfrak{N}, \nu)$. When $D_2(\mathscr{K}) = L_2(\Omega, \mathfrak{M}, \mu)$ the boundedness of $\mathscr{K}$ follows from the Closed-Graph Theorem. ∎

For any normed linear space $X$, let $\mathfrak{B}(X)$ be the set of bounded linear operators from $(X, \|\cdot\|)$ to itself. Let $\mathscr{D}(\mathscr{K}^\dagger \circ \mathscr{K}) = \{u \in \mathscr{D}(\mathscr{K}): \mathscr{K}u \in \mathscr{D}(\mathscr{K}^\dagger)\}$ and say that $\mathscr{K}^\dagger \circ \mathscr{K} \in \mathfrak{B}(X)$ if $X \subset \mathscr{D}(\mathscr{K}^\dagger \circ \mathscr{K})$ and $\mathscr{K}^\dagger \circ \mathscr{K}$ is a bounded linear operator on $(X, \|\cdot\|)$. Similarly $\mathscr{D}(\mathscr{K} \circ \mathscr{K}^\dagger)$ is defined and we say that $K$ is symmetrical if $(\Gamma, \mathfrak{N}, \nu) = (\Omega, \mathfrak{M}, \mu)$ and $K(x, y) = K(y, x)$.

Now let $\mathfrak{X}(\Omega, \mathfrak{M}, \mu)$ (and similarly $\mathfrak{X}(\Gamma, \mathfrak{N}, \nu)$) denote the set of all normed linear spaces $(X, \|\cdot\|)$ of functions on $\Omega$ with the following properties:

(a) if $f \in X$ then $f$ is $\mathfrak{M}$-measurable;

(b) if $\{f_n\}$ is a Cauchy sequence in $(X, \|\cdot\|)$ and $0 \leqslant f_n \leqslant f_{n+1}$ for all $n$ sufficiently large, then, on every $\sigma$-finite set $U \subset \Omega$, $\{f_n(x)\}$ is bounded above $\mu$-almost-everywhere;

(c) for each $\sigma$-finite set $U \subset \Omega$ there exists $f \in X$ with $f \geqslant 0$ $\mu$-almost-everywhere on $\Omega$ and $f > 0$ $\mu$-almost-everywhere on $U$.

Our most general result is the following observation about the behaviour of $\mathscr{K}$ on various spaces. Since $(\Gamma, \mathfrak{N}, \nu)$ and $(\Omega, \mathfrak{M}, \mu)$ are interchangeable, similar results hold for $\mathscr{K}^\dagger \circ \mathscr{K}$ and for $\mathscr{K}^\dagger \circ \mathscr{K}$; we quote only for the former. Let $\|\cdot\|_2$ denote the norm of a bounded linear operator from $L_2(\Omega, \mathfrak{M}, \mu)$ to $L_2(\Gamma, \mathfrak{N}, \nu)$ or vice versa.

PROPOSITION 1.2. *$\mathscr{K}$ is a bounded linear operator from $L_2(\Omega, \mathfrak{M}, \mu)$ to $L_2(\Gamma, \mathfrak{N}, \nu)$ if and only if there exists $X \in \mathfrak{X}(\Omega, \mathfrak{M}, \mu)$ such that $\mathscr{K}^\dagger \circ \mathscr{K} \in \mathfrak{B}(X)$. Moreover, for all such $X$,*

$$\|\mathscr{K}\|_2^2 \leqslant \|\mathscr{K}^\dagger \circ \mathscr{K}\|_{\mathfrak{B}(X)}$$

*and*

$$\|\mathscr{K}\|_2^2 = \|\mathscr{K}^\dagger\|_2^2 = \|\mathscr{K}^\dagger \circ \mathscr{K}\|_{\mathfrak{B}(L_2(\Omega, \mathfrak{M}, \mu))} = \|\mathscr{K} \circ \mathscr{K}^\dagger\|_{\mathfrak{B}(L_2(\Gamma, \mathfrak{N}, \nu))}.$$

*The operators $\mathscr{K}$ and $\mathscr{K}^\dagger$ are then Hilbert-space adjoints of each other.*

*Proof.* Suppose that $\mathscr{K}: L_2(\Omega, \mathfrak{M}, \mu) \to L_2(\Gamma, \mathfrak{N}, \nu)$ is a bounded linear operator. Then for every $u \in L_2(\Omega, \mathfrak{M}, \mu)$ and $v \in L_2(\Gamma, \mathfrak{N}, \nu)$,

$$\left| \int_\Omega (u \mathscr{K}^\dagger v) \, d\mu_y \right| \leqslant \int_\Omega |u(y)| \left\{ \int_\Gamma K(x, y) \, |v(x)| \, d\nu_x \right\} d\mu_y$$
$$= \int_\Gamma |v(x)| \left\{ \int_\Omega K(x, y) \, |u(y)| \, d\mu_y \right\} d\nu_x,$$

by Tonelli's Theorem,

$$= \int_\Gamma |v(x)| \, (\mathscr{K} \, |u|)(x) \, d\nu_x \leqslant \|\mathscr{K}\|_2 \, \|u\|_{L_2(\Omega, \mathfrak{M}, \mu)} \, \|v\|_{L_2(\Gamma, \mathfrak{N}, \nu)}.$$

Hence, by the Riesz Representation Theorem, $\mathscr{K}^\dagger: L_2(\Gamma, \mathfrak{N}, \nu) \to L_2(\Omega, \mathfrak{M}, \mu)$ is a bounded linear operator, and

$$\|\mathscr{K}^\dagger\|_2 \leqslant \|\mathscr{K}\|_2.$$

By symmetry, also, $\|\mathscr{K}^\dagger\|_2 \geqslant \|\mathscr{K}\|_2$. Therefore $\mathscr{K}^\dagger \circ \mathscr{K}$ is a bounded linear operator on $L_2(\Omega, \mathfrak{M}, \mu)$ and

$$\|\mathscr{K}^\dagger \circ \mathscr{K}\|^2_{\mathfrak{B}(L_2(\Omega, \mathfrak{M}, \mu))} = \|\mathscr{K}\|_2 \, \|\mathscr{K}^\dagger\|_2.$$

Since $L_2(\Omega, \mathfrak{M}, \mu) \in \mathfrak{X}(\Omega, \mathfrak{M}, \mu)$, this proves the "only if" part.

Now suppose that $X \in \mathfrak{X}(\Omega, \mathfrak{M}, \mu)$ and that $\mathscr{K}^\dagger \circ \mathscr{K}: X \to X$ is a bounded linear operator. Let $u \in L_2(\Omega, \mathfrak{M}, \mu)$ and let $f \in X$ be such that $f \geqslant 0$ on $\Omega$ and $f > 0$ $\mu$-almost-everywhere on the $\sigma$-finite set $U = \{x \in \Omega : u(x) \neq 0\}$. (Such a function exists by (c).) Choose $\lambda$ with

$$\lambda > \|\mathscr{K}^\dagger \circ \mathscr{K}\|_{\mathfrak{B}(X)}. \tag{1.1}$$

Then

$$g_n = \sum_{k=0}^{n} \lambda^{-k} (\mathscr{K}^\dagger \circ \mathscr{K})^k \, (f)$$

defines a Cauchy sequence $\{g_n\}$ in $(X, \|\cdot\|)$ with $g_n(x) \leqslant g_{n+1}(x)$ for $x \in \Omega$ and $g_n(x) > 0$ for $\mu$-almost-all $x \in U$. By (b), $g_n(x)$ converges monotonically $\mu$-almost-everywhere on $U$ to a positive finite limit, $g(x) > 0$ say, $x \in U$. By the Monotone-Convergence Theorem,

$$(\mathscr{K}^\dagger \circ \mathscr{K}) \, g(x) = \lambda g(x) - f(x) < \lambda g(x) \qquad \text{for } \mu\text{-almost-every} \quad x \in U \tag{1.2}$$

and for all $\lambda$ satisfying (1.1). Hence, for $x \in \Gamma$,

$$
\begin{aligned}
|\mathscr{K} u(x)|^2 &= \left| \int_\Omega K(x, y)\, u(y)\, d\mu_y \right|^2 \\
&= \left| \int_U \{K(x, y)\, g(y)\}^{1/2} K(x, y)^{1/2} \frac{u(y)}{g(y)^{1/2}}\, d\mu_y \right|^2 \\
&\leqslant \left\{ \int_\Omega K(x, y)\, g(y)\, d\mu_y \right\} \left\{ \int_U K(x, y) \frac{u(y)^2}{g(y)}\, d\mu_y \right\}, \qquad (1.3)
\end{aligned}
$$

by the Cauchy–Schwarz inequality. Therefore, by Tonelli's Theorem and (1.2),

$$
\begin{aligned}
\|\mathscr{K} u\|^2_{L_2(\Gamma, \mathfrak{N}, \nu)} &\leqslant \int_U \{\mathscr{K}^\dagger \circ \mathscr{K}(g)(y)\} \{u(y)^2/g(y)\}\, d\mu_y \\
&\leqslant \lambda \int_\Omega u(y)^2\, d\mu_y = \lambda\, \|u\|^2_{L_2(\Omega, \mathfrak{M}, \mu)}. \qquad (1.4)
\end{aligned}
$$

This completes the proof of the proposition. ▌

Corollary 1.3. *If $K$ is symmetrical then $\mathscr{K} \in \mathfrak{B}(L_2(\Omega, \mathfrak{M}, \mu))$ if and only if $\mathscr{K} \in \mathfrak{B}(X)$ for some $X \in \mathfrak{X}(\Omega, \mathfrak{M}, \mu)$.*

*Proof.* If $\mathscr{K} \in \mathfrak{B}(L_2(\Omega, \mathfrak{M}, \mu))$ then there is nothing to prove since $L_2(\Omega, \mathfrak{M}, \mu) \in \mathfrak{X}(\Omega, \mathfrak{M}, \mu)$. If, on the other hand, $\mathscr{K} \in \mathfrak{B}(X)$ then, by the symmetry, $\mathscr{K} = \mathscr{K}^\dagger$ and $\mathscr{K}^\dagger \circ \mathscr{K} = \mathscr{K}^2 \in \mathfrak{B}(X)$. The result now follows from Proposition 1.2. ▌

*Remarks.* Clearly an analogous proposition holds when $K\colon \Gamma \times \Omega$ takes values in the set of non-negative $n \times n$ matrices an $X$ is a space of vector-valued functions. The simple examples below have natural extensions to that context.

If $K$ is symmetrical and $\mathscr{K} \in \mathfrak{B}(L_2(\Omega, \mathfrak{M}, \mu))$, then $(X, \|\cdot\|)$ can be chosen to be any linear subspace of $L_2(\Omega, \mathfrak{M}, \mu)$, no matter how small, provided it is invariant under $\mathscr{K}^\dagger \circ \mathscr{K}$ and contains functions which are positive $\mu$-almost-everywhere on $\sigma$-finite sets.

If $(\Omega, \mathfrak{M}, \mu)$ and $(\Gamma, \mathfrak{N}, \nu)$ denote Lebesgue measure on open subsets of $\mathbb{R}^N$, the result says that if $\mathscr{K}$ does not map $L_2(\Omega, \mathfrak{M}, \mu)$ boundedly to $L_2(\Gamma, \mathfrak{N}, \nu)$ then $\mathscr{K}^\dagger \circ \mathscr{K}$ cannot act as a bounded linear operator on most familiar function spaces, such as $C^{k, \alpha}(\Omega)$, $L_p(\Omega)$, $W^{m, p}(\Omega)$, even when endowed with suitably weighted or incomplete norms. (When $K$ is symmetrical, this conclusion is particularly striking).

EXAMPLE Suppose that $\Gamma = \Omega = \{1, ..., n\}$ with counting measure and $X_p = \mathbb{R}^n$ with the usual $l_p$-norm, $1 \leqslant p \leqslant \infty$. If $M$ is a positive $n \times n$ real matrix and $M^\dagger$ is its transpose, then for all $p \geqslant 1$ Proposition 1.2 gives the familiar result that

$$\begin{aligned}\|M\|^2_{\mathfrak{B}(X_2)} &\leqslant \|M\|_{\mathfrak{B}(X_p)} \|M^\dagger\|_{\mathfrak{B}(X_p)} \\ &= \|M\|_{\mathfrak{B}(X_p)} \|M\|_{\mathfrak{B}(X_q)}, \qquad \text{when} \quad \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

PROPOSITION 1.4 (Schur's Condition [2, p. 22]). *Suppose that there exists a positive measurable function $h \in \mathscr{D}(\mathscr{K}^\dagger \circ \mathscr{K})$, but not necessarily in $L_2(\Omega, \mathfrak{M}, \mu)$, and a real number $\Lambda$ such that*

$$\mathscr{K}^\dagger \circ \mathscr{K} h(x) \leqslant \Lambda^2 h(x) \qquad \textit{for } \mu\textit{-almost-all} \quad x \in \Omega.$$

*Then $\mathscr{K} : L_2(\Omega, \mathfrak{M}, \mu) \to L_2(\Gamma, \mathfrak{N}, \nu)$ is a bounded linear operator with $\|\mathscr{K}\|_2 \leqslant \Lambda$. In particular, if $K$ is symmetrical and there exists a positive measurable function $h$ and a real number $\Lambda$ such that*

$$\mathscr{K} h(x) \leqslant \Lambda h(x) \qquad \textit{for } \mu\textit{-almost-all} \quad x \in \Omega,$$

*then $\mathscr{K} : L_2(\Omega, \mathfrak{M}, \mu) \to L_2(\Omega, \mathfrak{M}, \mu)$ is a bounded linear operator with $\|\mathscr{K}\|_2 \leqslant \Lambda$.*

*Proof.* Let $X$ denote the normed linear space of $\mathfrak{M}$-measurable functions $u$ such that $u/h$ is $\mu$-essentially bounded with

$$\|u\|_X = \|u/h\|_{L_\infty(\Omega, \mathfrak{M}, \mu)}.$$

Note that $h \in X$ and since $h > 0$ it follows that $X \in \mathfrak{X}(\Omega, \mathfrak{M}, \mu)$. Also $\|\mathscr{K}^\dagger \circ \mathscr{K}\|_{\mathfrak{B}(X)} \leqslant \Lambda^2$. The result follows from the theorem, and the case when $K$ is symmetrical is immediate from the Corollary. ∎

*Remark.* Suppose in Proposition 1.4 that $(\Omega, \mathfrak{M}, \mu) = (\Gamma, \mathfrak{N}, \nu)$ and that $h \notin L_2(\Omega, \mathfrak{M}, \mu)$. Then $\Lambda$ is not an eigenvalue of $\mathscr{K}$ on $L_2(\Omega, \mathfrak{M}, \mu)$. To see this, suppose that $\Lambda$ is an eigenvalue of $\mathscr{K}$ on $L_2(\Omega, \mathfrak{M}, \mu)$ with eigenfunction $u \in L_2(\Omega, \mathfrak{M}, \mu)$. Then equality holds in (1.3) and (1.4) with $\lambda = \Lambda^2$ and $h = g$. Therefore, by the criterion for equality in the Cauchy–Schwarz inequality, $u$ is a scalar multiple of $h$, which is a contradiction.

Suppose additionally that $K$ is symmetrical in Proposition 1.4 and that there exists a positive measurable function $h$ and a real number $\Lambda$ such that

$$\mathscr{K} h(x) \leqslant \Lambda h(x) \qquad \text{for } \mu\text{-almost-all} \quad x \in \Omega.$$

Let $(Y_p, \|\cdot\|_p)$ be the Banach space of measurable functions $u$ on $\Omega$ with

$$\|u\|_p^p = \int_\Omega \frac{|u|^p}{h^{p-2}}\, d\mu, \qquad 1 \leqslant p < \infty.$$

Then $\mathscr{K} \in \mathfrak{B}(Y_p)$. (The case $p=2$ is part of Proposition 1.4.) This is immediate from Hölder's inequality as follows.

$$\begin{aligned}
|\mathscr{K}(|u|)(x)|^p &= \left\{\int_\Omega \{K(x, y)\, h(y)\}^{(p-1)/p} \left\{K(x, y)^{1/p} \frac{|u(y)|}{h(y)^{(p-1)/p}}\right\} d\mu_y\right\}^p \\
&\leqslant \left\{\int_\Omega K(x, y)\, h(y)\, d\mu_y\right\}^{(p-1)} \left\{\int_\Omega K(x, y) \frac{|u(y)|^p}{h(y)^{p-1}}\, d\mu_y\right\} \\
&\leqslant \Lambda^{(p-1)} h(x)^{(p-1)} \left\{\int_\Omega K(x, y) \frac{|u(y)|^p}{h(y)^{p-1}}\, d\mu_y\right\}.
\end{aligned}$$

Therefore

$$h(x)^{(2-p)}\, |\mathscr{K}(|u|)(x)|^p \leqslant \Lambda^{(p-1)} h(x) \left\{\int_\Omega K(x, y) \frac{|u(y)|^p}{h(y)^{p-1}}\, d\mu_y\right\}.$$

The result now follows by integrating both sides over $\Omega$ using the symmetry of $K$. In particular, if $h$ is a constant (that is, $\mathscr{K} \in \mathfrak{B}(L_\infty(\Omega, \mathfrak{M}, \mu))$), we find that $\mathscr{K} \in \mathfrak{B}(L_p(\Omega, \mathfrak{M}, \mu))$, $1 \leqslant p < \infty$.

Example (Chisholm and Everitt [1]). Let $(\Omega, \mathfrak{M}, \mu) = (\Gamma, \mathfrak{N}, \nu)$ denote the Lebesgue measure spaces on $(0, \infty)$, and let $u$ and $v$ be functions on $(0, \infty)$ such that, for every $x > 0$,

$$R(x) = \int_0^x u(y)^2\, dy < \infty, \qquad S(x) = \int_x^\infty v(y)^2\, dy < \infty.$$

Define an operator $\mathscr{K}$ via

$$\mathscr{K} f(x) = v(x) \int_0^x u(y)\, f(y)\, dy.$$

(The well-known Hardy operator is the case when $v(x) \equiv 1/x$ and $u(\cdot) \equiv 1$).

A theorem of Chisholm and Everitt [1] states that $\mathscr{K}$ is bounded on $L_2(0, \infty)$ if and only if

$$A = \sup_{x>0} \{R(x)\, S(x)\}^{1/2} < \infty,$$

and then $\|\mathscr{K}\|_2 \leqslant 2A$. (Equality holds for the Hardy operator).

Let us see how to prove the "if" part by our method, assuming (without loss of generality) that $u > 0$ almost everywhere on $(0, \infty)$. Define

$$h(y) = R(y)^{-1/2} u(y).$$

Then

$$\mathscr{K} h(x) = v(x) \int_0^x R(y)^{-1/2} \, dR(y) = 2v(x) \, R(x)^{1/2} \leqslant 2Av(x) \, S(x)^{-1/2},$$

and

$$\mathscr{K}^{\dagger} \circ \mathscr{K} h(y) = u(y) \int_y^{\infty} h(x) \, v(x) \, dx \leqslant -2Au(y) \int_y^{\infty} S(x)^{-1/2} \, dS(x)$$

$$= 4Au(y) \, S(y)^{1/2} \leqslant 4A^2 u(y) \, R(y)^{-1/2} = 4A^2 h(y).$$

Hence, by Proposition 1.4, $\|\mathscr{K}\| \leqslant 2A$.

Now we observe that Proposition 1.4 is sharp for $\sigma$-finite measure spaces.

PROPOSITION 1.5. *Suppose that $(\Omega, \mathfrak{M}, \mu)$ is $\sigma$-finite and $\mathscr{K}$ is a bounded linear operator from $L_2(\Omega, \mathfrak{M}, \mu)$ into $L_2(\Gamma, \mathfrak{N}, v)$. Then*

$$\|\mathscr{K}\|_2 = \inf \{ |\Lambda| : \mathscr{K}^{\dagger} \circ \mathscr{K}(g) \leqslant \Lambda^2 g : g > 0 \ \mu\text{-almost-everywhere on } \Omega \}$$

$$= \inf \{ |\Lambda| : \mathscr{K}^{\dagger} \circ \mathscr{K}(g) \leqslant \Lambda^2 g : g \in L_2(\Omega, \mathfrak{M}, \mu),$$

$$g > 0 \ \mu\text{-almost-everywhere on } \Omega \}.$$

*Proof.* We have $\|\mathscr{K}^{\dagger}\| = \|\mathscr{K}\|$ and $\|\mathscr{K}^{\dagger} \circ \mathscr{K}\| = \|\mathscr{K}\|_2^2$. Choose $\lambda$ with $\lambda > \|\mathscr{K}\|_2^2$. Choose $f > 0$ $\mu$-almost-everywhere on $\Omega$ (possible because $(\Omega, \mathfrak{M}, \mu)$ is $\sigma$-finite) and let $g$ be defined as in the second part of the proof of Proposition 1.2. As there it follows that

$$(\mathscr{K}^{\dagger} \circ \mathscr{K}) \, g(x) = \lambda g(x) - f(x) < \lambda g(x) \qquad \text{for } \mu\text{-almost-every} \quad x \in U.$$

We can choose $f \in L_2(\Omega, \mathfrak{M}, \mu)$; and then we shall have $g \in L_2(\Omega, \mathfrak{M}, \mu)$. This proves the proposition. ∎

Note that this proposition means that estimating the norm of $\mathscr{K}^{\dagger} \circ \mathscr{K}$, and consequently of $\mathscr{K}$, is always just a case of finding good test functions which, significantly, need not be square-integrable.

## 2. AN EXAMPLE FROM WIENER–HOPF THEORY

Let $\beta \in (0, \infty)$. Use the notations

$$s(x) = \sin \tfrac{1}{2}\pi\, |x|, \qquad c(x) = \cos \tfrac{1}{2}\pi\, |x|,$$

$$S(y) = \sinh(\tfrac{1}{2}\pi y/\beta), \qquad C(y) = \cosh(\tfrac{1}{2}\pi y/\beta).$$

Let $\alpha \in (0, 1)$ be such that $\beta = c(\alpha)/s(\alpha)$, and consider the kernel

$$K_\alpha(x, y) = \frac{s(\alpha)}{\beta} \left\{ \frac{s(x)^\alpha\, S(y)^{1-\alpha}\, C(y)}{s(x)^2 + S(y)^2} \right\}, \qquad -1 < x < 0, \qquad 0 < y < \infty. \tag{2.1}$$

Let $(\Omega, \mathfrak{M}, \mu)$ denote $(0, \infty)$ with the usual Lebesgue measure and let $(\Gamma, \mathfrak{N}, \nu)$ represent $(-1, 0)$ with the $\sigma$-algebra of Lebesgue measurable sets where $\nu$ is Lebesgue measure multiplied by the constant $\beta^2$. Then

$$\mathscr{K}_\alpha u(x) = \int_0^\infty K_\alpha(x, y)\, u(y)\, dy \qquad \text{and} \qquad \mathscr{K}_\alpha^\dagger v(y) = \beta^2 \int_{-1}^0 K_\alpha(x, y)\, v(x)\, dx$$

where $dx$ and $dy$ denote the usual Lebesgue measures.

It is worth remarking now that the operator $\mathscr{K}_\alpha \mathscr{K}_\alpha^\dagger$ on $L_2(\Gamma, \mathfrak{N}, \nu)$ is never compact. One can easily produce a sequence $(w_n)$ of functions of norm 1 such that $(\mathscr{K}_\alpha \mathscr{K}_\alpha^\dagger w_n)$ can have no convergent subsequence. For example, one can take $w_n$ to be the normalised version of $x \mapsto x^{\eta_n}$, where the $\eta_n$ decrease to $-\frac{1}{2}$.

A Discussion which sets this example in context is given after the proof of the following proposition.

PROPOSITION 2.1. *When $\beta > 1$, equivalently, when $\alpha \in (0, \frac{1}{2})$,*

$$\|\mathscr{K}_\alpha\|_2 = \|\mathscr{K}_\alpha^\dagger\|_2 = \frac{1}{\sqrt{2}}.$$

*When $\beta \leqslant 1$, equivalently, when $\alpha \in [\frac{1}{2}, 1)$,*

$$\|\mathscr{K}_\alpha\|_2 = \|\mathscr{K}_\alpha^\dagger\|_2 = \frac{\sqrt{2\beta}}{1+\beta} = \sqrt{\frac{\sin(\pi\alpha)}{1 + \sin(\pi\alpha)}}.$$

*Proof.* For any $\gamma$, let

$$u_\gamma(y) = S(y)^{\gamma-1}, \quad y \in (0, \infty) \qquad \text{and} \qquad v_\gamma(x) = s(x)^{\gamma-1}, \quad x \in (-1, 0).$$

We first prove that if $\alpha-1<\gamma<\alpha+1$, then

$$\mathscr{K}_\alpha u_\gamma(x)=\frac{s(\alpha)}{c(\gamma-\alpha)}\, v_\gamma(x). \tag{2.2}$$

To see this, note that

$$\begin{aligned}
&\int_0^\infty K_\alpha(x, y)\, u_\gamma(y)\, dy \\
&\quad=\int_0^\infty K_\alpha(x, y)\, S(y)^{\gamma-1}\, dy \\
&\quad=\frac{2}{\pi}\, s(\alpha)\, s(x)^\alpha \int_0^\infty \frac{S^{\gamma-\alpha}}{s(x)^2+S^2}\, dS, \qquad \text{(since } S'(y)=(\pi/2\beta)\, C(y))\\
&\quad=\frac{1}{\pi}\, s(\alpha)\, s(x)^{\gamma-1} \int_0^\infty \frac{r^{(1+\gamma-\alpha)/2-1}}{1+r}\, dr \qquad \text{(where } r=S^2/s(x)^2)\\
&\quad=\frac{s(\alpha)\, s(x)^{\gamma-1}}{s(1+\gamma-\alpha)}=\frac{s(\alpha)}{c(\gamma-\alpha)}\, v_\gamma(x).
\end{aligned}$$

Note that for $\gamma>\frac{1}{2}$, $v_\gamma \in L_2(\Gamma, \mathfrak{N}, \nu)$ and $u_\gamma \in L_2(\Omega, \mathfrak{M}, \mu)$. Therefore if $\alpha-1<\gamma<\alpha+1$ and $\gamma>\frac{1}{2}$ it follows from (2.2) that $\|\mathscr{K}_\alpha\|_2$ (possibly infinite) satisfies the inequality

$$\|\mathscr{K}_\alpha\|_2 \geqslant \frac{s(\alpha)}{c(\gamma-\alpha)} \frac{\|v_\gamma\|_{L_2(\Gamma, \mathfrak{N}, \nu)}}{\|u_\gamma\|_{L_2(\Omega, \mathfrak{M}, \mu)}}.$$

However, the simple anti-clockwise contour integral $\int (\sin z)^{-p}\, dz$ taken down the positive imaginary axis, along the real interval $[0, \frac{1}{2}\pi]$, and up the half-line $\{\frac{1}{2}\pi\} \times (0, \infty)$, gives that

$$\int_0^{(1/2)\pi} \frac{dx}{\sin^p x}=\sin(\tfrac{1}{2} p\pi) \int_0^\infty \frac{dy}{\sinh^p y} \qquad \text{for} \quad p<1.$$

Therefore for all $\gamma>\frac{1}{2}$,

$$\frac{\|v_\gamma\|_{L_2(\Gamma, \mathfrak{N}, \nu)}}{\|u_\gamma\|_{L_2(\Omega, \mathfrak{M}, \mu)}}=\sqrt{\beta \sin((1-\gamma)\, \pi)}.$$

Since, for $\alpha \in (0, 1)$, $\gamma$ can be as close as we like to and greater than $\frac{1}{2}$, the conclusion is that

$$\|\mathscr{K}_\alpha\|_2 \geqslant \sqrt{\beta}\, \frac{s(\alpha)}{c(\frac{1}{2}-\alpha)}=\frac{\sqrt{2\beta}}{1+\beta}, \qquad \text{for all} \quad \alpha \in (0, 1), \tag{2.3}$$

with the understanding at this stage that the left-hand side may be infinite. To see that this bound is sometimes, but not always, sharp, we calculate $\mathscr{K}_\alpha^\dagger v_\gamma(y)$ using contour integration, as follows. First note that

$$\mathscr{K}_\alpha^\dagger v_\gamma(y) = \frac{2\beta s(\alpha)\, S(y)^{1-\alpha}\, C(y)}{\pi}\, I(y), \tag{2.4}$$

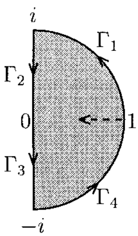
where

$$I(y) = \frac{\pi}{2}\int_{-1}^{0} \frac{s(x)^{\alpha+\gamma-1}}{s(x)^2 + S(y)^2}\, dx = \int_0^{(1/2)\pi} \frac{(\sin\theta)^{\alpha+\gamma-1}}{\sin^2\theta + S(y)^2}\, d\theta.$$

For a convenient notation, let $\delta = \alpha + \gamma - 1$, and for any non-zero complex number $z$, let

$$\phi(z) = -\tfrac{1}{2} i(z - z^{-1}).$$

Now observe that $\phi(z)^\delta$ can be defined for all non-zero $z$ which lie in the intersection of the unit disc and the right complex half-plane using the usual branch of log, because $\Im(\phi(z)) \geqslant 0$ for all such $z$. Let $\Gamma$ denote the union of the four contours with the anti-clockwise orientation



Contour Γ

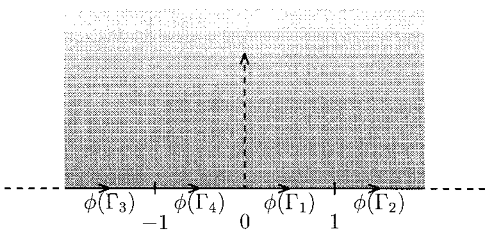
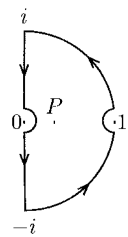


Image of half-disc under $\phi$



Contour $\Gamma_\epsilon$

implicit in the given parametrisations:

$$\Gamma_1 = \{e^{i\theta}: 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi\}, \qquad \Gamma_2 = \{iy: 1 \geqslant y \geqslant 0\},$$

$$\Gamma_4 = \{e^{i\theta}: -\tfrac{1}{2}\pi \leqslant \theta \leqslant 0\}, \qquad \Gamma_3 = \{iy: 0 \geqslant y \geqslant -1\}.$$

Now

$$\int_0^{(1/2)\pi} \frac{(\sin\theta)^\delta}{\sin^2\theta + S(y)^2}\, d\theta = \int_{\Gamma_1} \frac{\phi(z)^\delta}{\phi(z)^2 + S(y)^2}\, \frac{dz}{iz}$$

$$= (-1)^{-\delta} \int_{\Gamma_4} \frac{\phi(z)^\delta}{\phi(z)^2 + S(y)^2}\, \frac{dz}{iz},$$

since $\phi(\bar{z}) = -\phi(z)$ when $z \in \Gamma$. For the same reason,

$$\int_{\Gamma_2} \frac{\phi(z)^\delta}{\phi(z)^2 + S(y)^2} \frac{dz}{iz} + (-1)^{-\delta} \int_{\Gamma_3} \frac{\phi(z)^\delta}{\phi(z)^2 + S(y)^2} \frac{dz}{iz} = 0.$$

The integrand, $f(z)$ say, of these integrals has branch points at 0 and 1 in the complex plane where, for $z$ in the open unit disc and $y > 0$,

$$|f(z)| = O(|z|^{1-\delta}) \text{ as } z \to 0 \qquad \text{and} \qquad |f(z)| = O(|z-1|^\delta) \quad \text{as} \quad z \to 1.$$

Let $\Gamma_\varepsilon$ denote the contour obtained from $\Gamma$ by deforming it in small neighbourhoods of the branch points using arcs of circles with radius $\varepsilon$ centered at 0 and 1 lying in the unit disc. Then for $-1 < \delta < 2$ (that is, $\alpha + \gamma \in (0, 3)$)

$$\{1 + (-1)^\delta\} \int_{\Gamma_1} f(z)\, dz + \{1 - (-1)^\delta\} \int_{\Gamma_2} f(z)\, dz$$
$$= \int_\Gamma f(z)\, dz = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} f(z)\, dz. \tag{2.5}$$

Moreover, we note here that for $y > 0$ and $\delta < 2$,

$$\int_{\Gamma_2} f(z)\, dz = i \int_0^1 \frac{\{(t^2+1)/2t\}^\delta}{\{(t^2+1)/2t\}^2 + S(y)^2} \frac{dt}{t}$$
$$= iB(y) \qquad \text{where} \quad B(y) > 0. \tag{2.6}$$

The "symmetry" of things is nicely reflected in the fact that

$$B(y) = \frac{2\beta}{\pi} \int_0^\infty \frac{C(r)^\delta}{C(r)^2 + S(y)^2}\, dr.$$

Now, with $y = 2\beta Y/\pi$, the denominator of $f$ may be written

$$iz(\phi(z)^2 + S(y)^2) = iz(\phi(z)^2 + \sinh^2(Y))$$
$$= \frac{(z - e^{-Y})(z + e^{-Y})(z - e^Y)(z + e^Y)}{4iz},$$

and therefore, for $\varepsilon > 0$ sufficiently small, there is only one singularity of $f$ inside $\Gamma_\varepsilon$, namely a pole $P$ at $e^{-Y}$ with residue

$$\frac{1}{2i} \frac{S(y)^{\delta-1}}{C(y)} i^\delta = \frac{1}{2i} \frac{S(y)^{\delta-1}}{C(y)} e^{(1/2)\, i\delta\pi}. \tag{2.7}$$

Combining (2.5), (2.6), and (2.7) with the Residue Theorem, we find for $\delta \in (-1, 2)$ that

$$\begin{aligned} I(y) &= \int_{\Gamma_1} f(z)\,dz \\ &= \lim_{\varepsilon \to 0} \left\{ \frac{1}{1+(-1)^\delta} \int_{\Gamma_\varepsilon} f(z)\,dz \right\} - \left\{ \frac{1-(-1)^\delta}{1+(-1)^\delta} B(y)\, i \right\} \\ &= \pi \{1+(-1)^\delta\}^{-1} \frac{S(y)^{\delta-1}}{C(y)} e^{(1/2)\, i\delta\pi} - \tan(\pi\delta/2)\, B(y) \\ &= \pi \frac{S(y)^{\delta-1}}{C(y)} \frac{e^{(1/2)\, i\delta\pi}}{1+e^{i\delta\pi}} - \tan(\pi\delta/2)\, B(y) \\ &= \frac{\pi}{2\cos(\frac{1}{2}\delta\pi)} \frac{S(y)^{\delta-1}}{C(y)} - \tan(\pi\delta/2)\, B(y). \end{aligned} \tag{2.8}$$

*Case* 1. *First consider the case of* $\beta > 1$, *equivalently*, $\alpha \in (0, \frac{1}{2})$; *and let* $\gamma = 1 - \alpha$. Then $\delta = 0$, and

$$I(y) = \frac{\pi}{2S(y)\, C(y)} \qquad \text{and} \qquad \mathscr{K}_\alpha^\dagger v_\gamma(y) = \beta s(\alpha)\, u_\gamma(y).$$

Then from (2.2), (2.4) and the definition of $\beta$,

$$\mathscr{K}_\alpha \circ \mathscr{K}_\alpha^\dagger v_\gamma(x) = \frac{\beta s(\alpha)^2}{c(\gamma-\alpha)} v_\gamma(x) = \frac{s(\alpha)\, c(\alpha)}{s(2\alpha)} v_\gamma(x) = \frac{1}{2} v_\gamma(x).$$

Therefore, since $v_\gamma$ is square-integrable, because $\gamma > \frac{1}{2}$, $\|\mathscr{K}_\alpha\|_2 = \sqrt{\frac{1}{2}}$ when $\alpha \in (0, \frac{1}{2})$.

*Case* 2. *Now suppose that* $\alpha \in [\frac{1}{2}, 1)$. Then for any $\gamma$ with $\alpha + \gamma \in (1, 2]$ we have $\delta = \alpha + \gamma - 1 \in (0, 1]$ and therefore $\tan\, (\pi\delta/2)\, B(y) \geqslant 0$. For all such $\gamma$ and $\alpha$ we have shown in (2.4) and (2.8) that

$$\mathscr{K}_\alpha^\dagger v_\gamma(y) \leqslant \frac{\beta s(\alpha)}{c(\delta)} S(y)^{\delta-\alpha} = \frac{\beta s(\alpha)}{c(\alpha+\gamma-1)} S(y)^{\gamma-1} = \left\{ \frac{\beta s(\alpha)}{c(\alpha+\gamma-1)} \right\} u_\gamma(y),$$

and hence, from (2.2) and the definition of $\beta$,

$$\mathscr{K}_\alpha \circ \mathscr{K}_\alpha^\dagger v_\gamma(x) \leqslant \frac{s(\alpha)\, c(\alpha)}{c(\gamma-\alpha)\, c(\alpha+\gamma-1)} v_\gamma(x) = \frac{\sin \pi\alpha}{\sin \pi\gamma + \sin \pi\alpha} v_\gamma(x).$$

Since the coefficient of $v_\gamma$ on the right-hand side is minimised when $\gamma=\frac{1}{2}$, and since then $\frac{1}{2}+\alpha\in[1,2]$ we have proved in particular that

$$\mathscr{K}_\alpha\circ\mathscr{K}_\alpha^\dagger v_{1/2}(x)\leqslant\frac{\sin(\pi\alpha)}{1+\sin(\pi\alpha)}v_{1/2}(y)=\frac{2\beta}{(1+\beta)^2}v_{1/2}(y).$$

Therefore, by Proposition 1.2, $\|\mathscr{K}_\alpha\|_2\leqslant\sqrt{2\beta/(1+\beta)^2}$ when $\alpha\in[\frac{1}{2},1)$; and, because of (2.3), we must have equality here. This completes the proof. ∎

*Discussion of the Significance of $K_\alpha$*

Wiener–Hopf theory is concerned with "factorizing" certain operators. Williams [4] is a survey, but much better accounts are now possible, and (at least) one is being prepared

Let $E^+=(0,\infty)$, $E^-=[-1,0]$, $m^+=\text{Leb}$ on $E^+$, $m^-=\beta^2\times\text{Leb}$ on $E^-$. Let $E=E^+\cup E^-$, and let the measure $m$ on $E$ have restrictions $m^\pm$ to $E^\pm$. We have the canonical identification

$$\tilde{L}_2=\tilde{L}_2^+\oplus\tilde{L}_2^-,\qquad\text{where}\quad\tilde{L}_2=L_2(E,m),\qquad\tilde{L}_2^\pm=L_2(E^\pm,m^\pm).$$

The tildes help emphasise that the inner products on $\tilde{L}_2$ and $\tilde{L}^-$ are not the standard ones relative to Lebesgue measure. Let $I^\pm$ be the identity operator(s) on $\tilde{L}_2^\pm$.

The operator

$$\mathscr{A}=\frac{1}{2}\frac{d^2}{dy^2}\quad(y>0),\qquad\mathscr{A}=\frac{1}{2\beta^2}\frac{d^2}{dx^2}\quad(-1<x<0),$$

acting on

$$\tilde{L}_2\cap C^1[-1,\infty)\cap C^2(-b,0)\cap C^2(0,\infty)$$
$$\cap\{f\colon f'(-1)=0\}\cap\{f\colon\mathscr{A}f\in\tilde{L}_2\}$$

is essentially self-adjoint on $\tilde{L}_2$ (strictly self-adjoint if we interpret derivatives in the "absolute continuity" sense). Much of its importance derives from the fact that its closure generates a meaningful semigroup of self-adjoint contraction operators. Let $V$ be the sgn function on $\mathbb{R}$. The kernel $K=K_\alpha$ arises in the factorisation

$$J^{-1}V^{-1}\mathscr{A}J=\begin{pmatrix}\mathscr{G}_+ & 0\\ 0 & -\mathscr{G}_-\end{pmatrix},\qquad\text{where}\quad J=\begin{pmatrix}I^+ & \mathscr{K}^\dagger\\ \mathscr{K} & I^-\end{pmatrix}\tag{2.9}$$

where $\mathscr{G}_+$ is a negative-semi-definite self-adjoint operator on functions on $[0,\infty)$ relative to the inner product

$$\langle f,g\rangle_+=\langle f,(I-\mathscr{K}^\dagger\mathscr{K})\,g\rangle_{m^+},$$

the $\langle \cdot, \cdot \rangle_{m^+}$ on the right-hand side being the standard inner product for $\tilde{L}_2^+$. Analogously, $\mathscr{G}_-$ is a negative-semi-definite self-adjoint operator on functions on $[-1, 0]$ relative to the inner product

$$\langle f, g \rangle_- = \langle f, (I - \mathscr{K}\mathscr{K}^\dagger)\, g \rangle_{m^-},$$

the $\langle \cdot, \cdot \rangle_m^-$ on the right-hand side now being the standard inner product for $\tilde{L}_2^-$. It is immediately clear that we need the operators $(I - \mathscr{K}^\dagger \mathscr{K})$ and $(I - \mathscr{K}\mathscr{K}^\dagger)$ to be invertible; and that is one key use of the fact that we have $\|\mathscr{K}\| \leqslant 2^{-1/2}$ for every $\alpha$. The operators $\mathscr{G}_+$ and $\mathscr{G}_-$ generate contraction semigroups of self-adjoint operators (relative to the $\langle \cdot, \cdot \rangle_\pm$ inner products) which have probabilistic significance.

*Why does $K$ take the particular form in* (2.1)? For $\theta > 0$, the function $f_\theta$ on $[-1, \infty)$ with

$$f_\theta(r) = \begin{cases} f_\theta(0) \cos \theta r + \theta^{-1} \sin \theta r & \text{if} \quad r > 0, \\ f_\theta(0) \cosh \theta\beta(r+1) & \text{if} \quad -1 \leqslant r \leqslant 0, \end{cases}$$

$$\text{where} \quad f_\theta(0) = (\theta\beta \sinh \theta\beta)^{-1},$$

satisfies $\mathscr{A} f_\theta = -\frac{1}{2}\theta^2 V f_\theta$. Formally at least, since $-\mathscr{G}_-$ has only non-negative eigenvalues and $-\frac{1}{2}\theta^2$ is negative, $J^{-1} f_\theta$ must be 0 on $E^-$; in other words, we require that $K$ satisfy

$$f_\theta(x) = \int_0^\infty K(x, y)\, f_\theta(y)\, dy \qquad (-1 \leqslant x < 0). \tag{2.10}$$

This fact completely determines $K$—see London, McKean, Rogers and Williams [3] for the relevant existence and uniqueness theorem. Of course, one can verify by contour integration that our $K_\alpha$ satisfies (2.10).

Because one is dealing with generators of self-adjoint semigroups, the $L_2$ setting is in one sense natural. The self-adjointness corresponds to time-reversibility properties (some of which are rather mysterious) in the probability theory. In Williams [4], the whole setup is considered in terms of Dirichlet forms which are of course naturally linked to $L_2$ theory. One gets a nice overall picture, of which part is the "duality property" that $\mathscr{K}^\dagger$, the Hilbert-space adjoint of $\mathscr{K}$, is the correct "half-winding" operator for appearance in the factorisation (2.9).

Other settings for the theory play an equally important part, however. We can regard $\mathscr{A}$ (with modified domain) as the generator of a semigroup of operators of norm 1 on the space $C_0(E)$ of bounded continuous functions on $E$ tending to 0 at infinity. One can also consider semigroups on $L_1$ and $L_\infty$. On these spaces, it is often the case that one loses invertibility of operators $I - \mathscr{K}^\dagger \mathscr{K}$ and $I - \mathscr{K}\mathscr{K}^\dagger$; and, from that point of view, the factorisation (2.9) is then much less satisfactory! The whole picture is a rather complicated one; and one needs to bear all functional-analytic settings in mind.

A puzzling feature of central importance relates to problems of uniform integrability. Williams and Marles [5] have begun an explanation of why a large class of Wiener–Hopf kernels are likely to have $L_2$ norms no greater than $2^{-1/2}$, a result depending on our Proposition 1.4. This present paper has sought to clarify this initially surprising result by determining certain $L_2$ norms precisely. In the example which we have been studying in this section, it is very plausible on probabilistic grounds that (as we have proved to be the case) $\|\mathscr{K}_\alpha\|_2$ is a non-decreasing function of $\beta$; and having already reached the magic number $2^{-1/2}$ when $\beta = 1$, the norm can go no higher. But, from some points of view, there are mysteries about why the norm drops below $2^{-1/2}$ for $\beta < 1$.

There is another aspect to all this. Suppose that for $b > 0$ we define a kernel on $[-b, 0] \times (0, \infty)$ via

$$K_{b,\alpha}(x, y) = b^{-1} K_\alpha(b^{-1}x, b^{-1}y).$$

It is not difficult to see that, for every $b$, $\mathscr{K}_{b,\alpha}$, considered as mapping $L_2([-b, 0), \beta^2 \times \mathrm{Leb})$ to $L_2^+$ has the same norm as our $\mathscr{K}_\alpha$. As $b \to \infty$, $K_{b,\alpha}(\cdot, \cdot)$ converges pointwise to the kernel $K_{\infty,\alpha}$ on $(-\infty, 0) \times (0, \infty)$, where

$$K_{\infty,\alpha}(x, y) = \frac{2\beta^\alpha |x|^\alpha y^{1-\alpha}}{\pi(\beta^2 x^2 + y^2)} \sin \tfrac{1}{2}\pi\alpha. \tag{2.11}$$

Now (see Williams and Marles [5], or via a change of variable, derive the result from the well-known result on the norm of convolution operators), this kernel has $L_2$ norm $(2\beta)^{1/2}(1+\beta)^{-1}$ for *every* $\beta$ in $(0, 1)$. So, for $\beta > 1$, as $b \to \infty$, there is a "Fatou" drop in the norm from $2^{-1/2}$ to this value. The similar phenomenon on finite intervals which was hinted at in the preceding paragraph is much more puzzling, however.

The kernel $K_{\infty,\alpha}$ satisfies $\mathscr{K}_{\infty,\alpha}\mathbf{1}^+ = \mathbf{1}^-$ and $\mathscr{K}^\dagger_{\infty,\alpha}\mathbf{1}^- = \mathbf{1}^+$, the $\mathbf{1}^\pm$ functions being the obvious constant functions equal to 1 on the respective spaces; so we lose the desirable invertibility of $I - \mathscr{K}^\dagger\mathscr{K}$ and its companion if we work on $L_\infty$.

Return now to the case which has occupied nearly all of this section. The clear probabilistic significance of the "eigenfunction" corresponding to $\gamma = 1 - \alpha$ is known from the Williams–Marles paper. According to the heuristic (proof to appear elsewhere) in that paper, we can (for every $\alpha$)

produce a positive function $p$ on $E^-$ with $\mathcal{K}_\alpha \mathcal{K}_\alpha^\dagger p \leqslant \frac{1}{2} p$ (whence $\|\mathcal{K}_\alpha \mathcal{K}_\alpha^\dagger\| \leqslant \frac{1}{2}$) by requiring that $p$ solve

$$\int_{E^-} p(x)\, g_\theta(x)\, m^-(dx) = 0$$

whenever $\theta > 0$ and $g_\theta$ is a function on $E^- = [-1, 0)$ with

$$\theta g_\theta(0) - g'_\theta(0) = 0, \qquad g'_\theta(-1) = 0, \qquad \tfrac{1}{2}\theta^2 g_\theta + \tfrac{1}{2}\beta^{-2} g''_\theta = 0.$$

We therefore require that

$$\int_{-1}^{0} p(x)\{\beta \cos \theta\beta x + \sin \theta\beta x\}\, dx = 0$$

$$\text{whenever} \quad \theta > 0 \qquad \text{and} \qquad \beta \tan \theta\beta = -1,$$

and we wish to verify that this holds for the solution found in this paper, namely, $p = v_\gamma$ when $\gamma = 1 - \alpha$. Recalling that $\beta = \cot \frac{1}{2}\pi\alpha$, we can prove the required result (after an $x \mapsto -x$ substitution) by considering the real part of the integral of

$$\frac{\exp\{i(\theta\beta z + \frac{1}{2}\pi\alpha)\}}{(\sin \frac{1}{2}\pi z)^\alpha}$$

along a contour which comes down the imaginary axis to 0, right along the real axis from 0 to 1, and up along the half-time $\{1\} \times [0, \infty)$.

The "supereigenfunctions" corresponding to $\gamma$ values greater than $\frac{1}{2}$ when $\alpha > \frac{1}{2}$ inevitably correspond to certain supermartingales; but their significance is rather hazy at this time.

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